Chaotic dynamics of an asymmetric circular billiard with ray splitting

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The ray evolution in a composite asymmetric circular billiard with ray refraction at the inner phase interface is studied. A quantitative criterion has been derived for the exponential angular widening of an arbitrary ray bundle as a necessary condition of the ray chaos. The possible reconstructions in the phase portrait of the mentioned billiard family in the phase space at changes in the controlling parameter (asymmetry) have been analyzed qualitatively. Lacunar features have been revealed and explained in the phase dynamics associated with the geometric thinning of the acceptable ray paths and with the total internal reflection of the rays from the billiard inner boundary. The results obtained are of importance in description of the light collection phenomena in composite detectors made of several different scintillation materials.

Исследована эволюция лучей в композитном асимметричном кольцевом биллиарде с преломлением лучей на внутренней границе раздела двух сред. Получен количественный критерий экспоненциального углового уширения произвольного пучка лучей как необходимое условие хаотизации лучевой картины. В фазовом пространстве проведен качественный анализ возможных перестроек фазового портрета указанного семейства биллиардов при изменении управляющего параметра (асимметрии). Обнаружены и объяснены лакунарные особенности фазовой динамики, связанные с геометрическим прореживанием допустимых маршрутов лучей и с эффектом полного внутреннего отражения лучей от внутренней границы биллиарда. Полученные результаты важны для описания явлений светосбора в комбинированных детекторах, образованных из нескольких разных сцинтилляционных материалов.

A billiard is a simplest physical system where the ray motion is defined by the geometric optics laws. Nevertheless, it is just the billiards that turned out to be a clear theoretical model for development of fundamental concepts in the determined chaos theory for the classical, statistical, and quantum physics [1-3].

The chaotic behavior of rays of an usual billiard may be caused by two mechanisms, namely, a continuous widening of an arbitrary ray bundle at its reflection from scattering (convex) boundary sections in Sinai billiards or defocusing of rays reflected from focusing (convex) boundary components, e.g., in Bunimovich stadium. In composite billiards with ray splitting, additional chaos mechanisms are to be expected [4, 5]. In connection with the ray "multiplication"at reflection from the boundary between media with different refraction indices (a single incident ray originates a reflected and a refracted ones), an indeterminacy in their paths arises. Each permissible route corresponds to a certain sequence in which a certain trajectory of the billiard visits different media. That route coded by a binary sequence (when the billiard consists of two media) can be either regular or chaotic. Respectively, the binary sequence can be periodic or aperiodic, in other words, a rational or irrational number within limits from zero to one could be correlated to it.

1. Asymmetrical billiard with ray splitting

The phase trajectories of a symmetrical analog [4, 5] of the system under investigation are regular while the mentioned concentric billiard is not ergodic. As to it, it is possible to speak only on a chaos as the indeterminancy in the visiting of differently refracting media by some trajectories, since there are routes or ray classes corresponding to the chaotic sequences, the plurality thereof having the cardinality of continuum. No other chaotic state arises in this case. However, at any arbitrarily small asymmetry, chaotic regions arise in the phase space of such a system that are associated with exponential scattering of initially close trajectories.

To describe the ray dynamics in the asymmetrical billiard, let the geometric-dynamic approach developed in [6] be used. The light intensity variation is not considered, similar to usual billiards. The ray dynamics is set using the maps transforming the incident ray (S_1, S_2) into the consequent one $(\overline{S_1}, \overline{S_2})$ reflected or refracted.

Both outer and inner boundaries of the billiard being considered are convex outwards. Therefore, angular coordinates are convenient to be selected as the ray ones. The coordinate origin lies at the center of the circle containing one end of the chosen ray (segment) of the trajectory. In such a consideration, the phase space consists of four sheets T^2, the coordinates of the segment start and end being laid on the coordinate axes of the sheets. Indexing the billiard outer boundary as 0 and the inner media interface as 1 (Fig. 1), then each of four sheets corresponds to the subspaces of rays with ends lying in pairs on the corresponding boundaries: $|11|, |00|, |01|, |10|$. The transitions from one sheet into other are described by five maps $F_{01}, F_{10}, F_{11}, F_{00}, \overline{F_{00}}$. Let the boundary number denote simultaneously the number of medium adjacent thereto. The number of the boundary limiting a medium is ascribed to the medium. In other words, the inner boundary and medium have the index 1 and the outer ones 0. Then the first figure in the map notations indicates the medium where the ray is directed to by the selected map and the second one, where it goes from. For example, the map F_{01} transforms the ray incident on the boundary 1 from the medium 1 into a refracted ray propagating in the medium 0. The map $\overline{F_{00}}$ leaves the ray within the medium 0 but, in contrast to F_{00}, it is reflected from the boundary 1. Note that evolution of all the rays in a billiard is defined completely by the set of above elementary maps. Using the variables (x, y) and $(\overline{x}, \overline{y})$ to define the ray coordinates prior to and after the reflection (refraction), respectively, let those maps be written as

$$F_{01}: \begin{cases} \overline{x} = f_{01}(x, y) \bmod 2\pi \\ \overline{y} = x \bmod 2\pi \end{cases} ; F_{00}: \begin{cases} \overline{x} = f_{00}(x, y) \bmod 2\pi \\ \overline{y} = x \bmod 2\pi \end{cases} ; F_{11}: \begin{cases} \overline{x} = f_{11}(x, y) \bmod 2\pi \\ \overline{y} = x \bmod 2\pi \end{cases} ;$$

$$F_{10}: \begin{cases} \overline{x} = f_{10}(x, y) \bmod 2\pi \\ \overline{y} = x \bmod 2\pi \end{cases} ; \overline{F_{00}}: \begin{cases} \overline{x} = \overline{f_{00}}(x, y) \bmod 2\pi \\ \overline{y} = x \bmod 2\pi \end{cases} \tag{1}$$

The explicit cumbersome expressions for f_{00}, f_{10}, f_{01}, f_{11} and $\overline{f_{00}}$ are omitted here.

2. Criterion of angular broadening of trajectory bundles

One possible causes for chaos in billiards is known to consist in the mechanism of the permanent ray scattering when any infinitesimal bundle diverges everywhere in the geometric space [7]. This is observed, e.g., in the Sinai billiards when the rays are reflected from the boundary that is concave inwards everywhere. In the phase space, the exponential divergence of initially close trajectories corresponds to that case. Thus, when studying billiards, it is useful to estimate the average rate of angular broadening at reflection from different components of the billiard boundary of a narrow or, in a specific case, plane-parallel ray bundle. Let such an estimation be found for the billiard under consideration by tracing the broadening of a narrow ray bundle after a single refraction.

Let the angular broadening be calculated for a ray bundle with the 1001 route, that is, its broadening at the return to the medium 1 after two reflections in the medium 0 (Fig. 1). The angular parameters

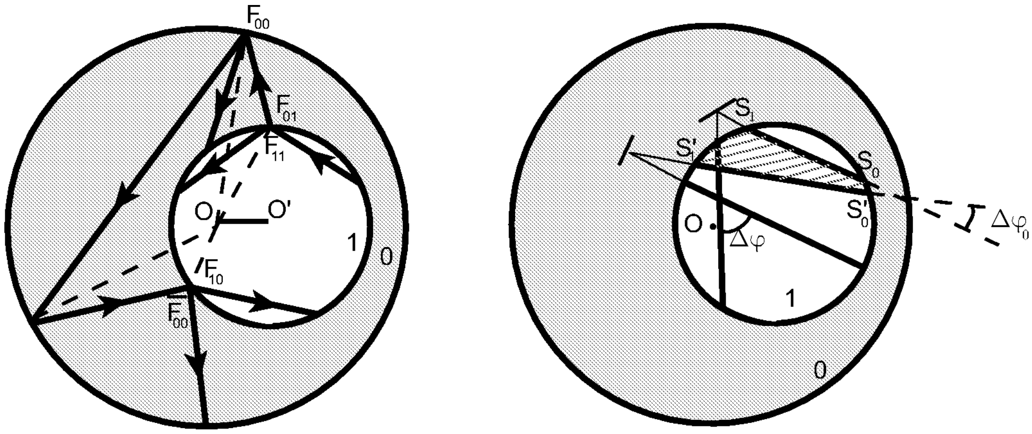

Fig. 1. Geometry of an asymmetric circular billiard with refraction (left). Points O and O' are centers of the outer and inner boundary, respectively. The distance therebetween is . Several elementary transitions of a billiard trajectory and the corresponding elementary maps F_{ij}. The angular broadening of a ray bundle subjected to refraction (right). The ray evolution due to refraction is simulated by reflection from an "effective boundary".

of the incident and thransmitted bundles are interrelated by the expression $\Delta\varphi = \tilde{K}\Delta\varphi_0$, where $\tilde{K} = \tilde{K}(S_1, S_2, \Delta\varphi_0)$ is the average broadening coefficient of the bundle being refracted; (S_1, S_2) are the coordinates of the carrier, e.g., of the central or one of limiting rays. Then for an infinitesimal bundle at $\Delta\varphi_0 \to 0$ the broadening coefficient is determined as

$$K = K(S_1, S_2) = \frac{\partial \Delta\varphi}{\partial \Delta\varphi_0} \tag{2}$$

If $K = 1$ at all the values of phase coordinates, then no chaos associated with refraction arises. On the other hand, if $K > 1$, the chaotic motion component may appear. At a small billiard asymmetry, it is convenient to seek for the relationship between $\Delta\varphi$ and $\Delta\varphi_0$ using the concept of effective boundary proposed in [4]. That concept makes in possible to disengage oneself from the refraction and propagation of each individual ray in a medium with other refractive index by substituting that multi-step process by reflection from a certain curve limiting the usual billiard (that curve is the effective boundary). It is of importance that it is possible to put into correspondence to each ray its effective boundary only, if the latter exists. For a symmetrical billiard, the effective boundary is a circle with a radius depending on the difference in angular coordinates $(S_2 - S_1)$ of the initial ray. For an asymmetrical billiard, the effective boundary concept can be kept only locally, because the position of the elementary area from which the effective reflection occurs replacing the ray propagation in the second medium will now depend on both coordinates S_1 and S_2. In general, a plurality of such elementary areas is not reduced to a smooth or even continuous curve.

When the asymmetry is small enough $\Delta \ll r$, the elementary area radius for the "effective boundary"which keeps its initial sense in this case is

$$\rho_{\text{eff}} = \rho_{\text{eff}}(S_1, S_2) = \frac{r\cos\left(\frac{S_2 - S_1}{2}\right)}{\cos\left[\frac{S_2 - S_1}{2} + \xi(S_1, S_2) - \arcsin\left(\frac{r + \Delta\cos S_2}{R}\sin\xi(S_1, S_2)\right)\right]}, \tag{3}$$

where the radius is measured from the center of the larger circle and the following function is introduced

$$\xi(S_1, S_2) = \arcsin\left[\frac{n_1}{n_0}\cos\left(\frac{rS_2 - rS_1 + \Delta(\sin S_2 - \sin S_1)}{2r}\right)\right] + \left(\frac{\Delta}{r}\right)\sin S_2$$

.

Let the incidence angles of the bundle-limiting rays onto the elementary area of the "effective boundary"be denoted as θ_1 and θ_2. Then the angular width of the transmitted bundle and the incident one are interrelated by the expression [4]:

$$\Delta\varphi = \Delta\varphi_0 + 2(\theta_1 - \theta_2)\ . \tag{4}$$

For a concentric symmetrical billiard $K = 1$, because the angles θ_1 and θ_2 depend only on the angular coordinate differences of the first or second bundle-limiting rays and are independent of the initial scatter $\Delta\varphi_0$. At small $\frac{\Delta}{r} \ll 1$, the broadening coefficient is defined by the following expression (its derivation is omitted)

$$K = 1 + \frac{4\Delta}{r}\left(A\cos S_2 - B\sin S_2\right)\ , \tag{5}$$

where

$$A = A\left(S_1, S_2\right) = 2 - \frac{\left(\frac{n_1}{n_0}\right)\sin\left(\frac{S_2-S_1}{2}\right)}{\sqrt{1-\left[\left(\frac{n_1}{n_0}\right)\sin\left(\frac{S_2-S_1}{2}\right)\right]^2}} + \frac{\left(\frac{rn_1}{Rn_0}\right)\sin\left(\frac{S_2-S_1}{2}\right)}{\sqrt{1-\left[\left(\frac{rn_1}{Rn_0}\right)\sin\left(\frac{S_2-S_1}{2}\right)\right]^2}}\ ;$$

$$B = B\left(S_1, S_2\right) = \cot\left(\frac{S_2 - S_1}{2}\right)\ .$$

It is easy to understand that at certain values (S_1, S_2), a phase region exists where the broadening coefficient exceeds one. For example, $K > 1$ for all $S_1 \in \left[0, \frac{\pi}{2}\right] \cup [\pi, 2\pi]$ if $S_2 \in \left[\frac{3\pi}{2}, 2\pi\right]$. Thus, a chaos associated with the ray scattering and transmission through a medium with different refraction index can be observed in the system. The features of that behavior can be studied in more detail using the phase portrait on the introduced map cascade (1).

3. Chaotic ray dynamics and phase transformations

Let the phase portrait and its transformations be considered for the family of billiards as the asymmetry parameter $\frac{\Delta}{r}$ increases. The symmetrical case $\Delta = 0$ was considered before [4]. In general case, the phase dynamics involves four phase space sheets $\Phi = \bigvee_{i,j} \mathbb{T}^2_{|ij|}$ joined together, where $i, j = (0, 1)$. To describe the dynamics unambiguously, the route of rays should be indicated along with their coordinates. Strictly speaking, when setting each initial ray by the coordinate pair (x, y), its route in to be preset. The latter is encoded by a binary sequence consisting of zeros and units according to the visiting sequence of the media 0 and 1 by the ray trajectory. Therefore, a real number s can be brought in correlation to each route within a unit section. Thus, the three quantities (x, y, s) set uniquely the initial condition for a billiard trajectory evolution.

Subdividing the plurality of all possible trajectories into equivalence classes according to the permissible routes (taking into account prohibition rules dictated by the billiard geometry), we can study the phase dynamics of the corresponding type trajectories. For the classes encoded by the simplest periodic sequences of $11\ldots$ or $00\ldots$ type, the problem is reduced to the dynamics study in a usual billiard. In the first case, it is a billiard in the inner circle; in the second one, the billiard with a doubly-connected boundary consisting of a pair of circles (the outer and inner boundaries of a composite billiard). Both mentioned dynamics types are associated only with the ray reflection from the boundaries of usual billiards. Let that type be referred to as the elementary or simple "reflection"dynamics. A novel non-trivial consideration is associated with the study of the trajectory class with periodical route [1001001...], the 100 series being the period. In this case, the repeated ray refraction at its single exit to the second medium followed by the return to the initial one. Thus, that dynamics type can be referred to as the elementary "refraction"dynamics. It is of a substantial interest for us. The phase portrait of trajectories with more complex routes, including non-periodic ones, is reduced to a composition of the simple "reflection"and "refraction"phase portraits, although the overlap sequence of such elementary portraits within the whole phase space may be complex enough and even a chaotic one.

The ray route cannot be indicated explicitly in all cases. For example, this is essentially impossible for chaotic sequences where there is no algebraic algorithm for reconstitution of an arbitrary section of the sequence proceeding from its end or initial section. But the ray route and its initial coordinates are matched always. That correlation is not mutually unique, since different routes may correspond to one and the same initial ray due to possible ray splitting. Nevertheless, the matching allows to use an alternative way to include the concept of the ray route into the dynamic description. At first, only the initial ray coordinates can be believed to be set. Then, having obtained the successive reflections and refractions of the originated rays according to the billiard dynamics, all its permissible routes are to be constructed and listed successively. Certain discriminated route types can be traced, e.g., the 1001001... "simple refraction"routes. In the route structure of a composite billiard so interpreted basing substantially on the usual intuition, the features of some prohibition rules (considered below in detail) can be explicitly found. The rules can restrict the class of permissible routes corresponding to specific pluralities of initial points within the system phase space.

At the simple refraction, the rays visit two media in the strict sequence $|11| \xrightarrow{F_{01}} |10| \xrightarrow{F_{00}} |01| \xrightarrow{F_{10}} |11| \to \dots$. The $|00|$ sheet is never visited. This is a demonstration of topology prohibition rule for the phase trajectories of the type under study. The phase portraits on each sheet can be considered separately. Let the consideration be restricted by the $\mathbb{T}^2_{|11|}$ sheet. The phase portrait thereon corresponds to the simple but repeated "refraction"dynamics. For a symmetric billiard, that sheet is stratified into single-dimensional invariant manifolds (curves). The ray propagation within a geometrical space corresponds to a fixed shift along those curves. The dynamics is described by superposition of maps $F = F_{10} \circ F_{00} \circ F_{01}$:

$$F : \begin{cases} \overline{x} = x + D\left(x_0 - y_0\right) \bmod 2\pi \\ \overline{y} = y + D\left(x_0 - y_0\right) \bmod 2\pi \end{cases},$$

where $D\left(x_0 - y_0\right) = \left[2\arcsin\left(\frac{n_1}{n_0}\cos\left(\frac{x_0-y_0}{2}\right)\right) - 2\arcsin\left(\frac{r}{R}\frac{n_1}{n_0}\cos\left(\frac{x_0-y_0}{2}\right)\right) + \left(x_0 - y_0\right)\right]$ is the motion integral for the symmetrical billiard. In the case of asymmetry, the map takes the form

$$F = \begin{cases} \overline{x} = x + D\left(x_0 - y_0\right) + \left(\frac{\Delta}{r}\right)\alpha_1\left(x, y\right) \bmod 2\pi \\ \overline{y} = y + D\left(x_0 - y_0\right) + \left(\frac{\Delta}{r}\right)\alpha_2\left(x, y\right) \bmod 2\pi \end{cases}, \tag{6}$$

where functions $\alpha_1\left(x, y\right)$ and $\alpha_2\left(x, y\right)$ in general case depend on both coordinates x and y and differ from one another.

The qualitative consideration of the map (6) phase portrait at increasing $\frac{\Delta}{r}$ shows that the invariant straight lines of undisturbed billiard are distorted at first, the distortion being maximum near the tore diagonal (Fig. 2). Furthermore, the stability degeneration is withdrawn. (The phase trajectories of an undisturbed billiard are characterized by zero stability, because both eigenvalues of the Jacobi matrix for the linearized map are equal to unity). The disturbed trajectories are subdivided into the hyperbolic trajectory class far from the diagonal and elliptical orbits around elliptical points arising at the asymmetry. At first, there are two immobile points on the diagonal with coordinates $(0, 0)$ and $(2\pi, 2\pi)$ that are joined together at the tore. Therewith, the stability islands that are well distinct at low asymmetry (Fig. 2) are encompassed by a separatrix having two loops closed in the hyperbolic fixed point (π, π) on the tore diagonal. The trajectories of the hyperbolic motion component are pressed to that unstable point. A chaotic zone (homoclinic structure) arises around that separatrix where the invariant curves are broken up. As the asymmetry increases, the near separatrix chaotic zone increases, too; resonances of ever-higher order appear surrounded by chaotic layers. The resonances become progressively overlapped and the near-diagonal phase space is filled with a chaotic ocean. At large deformations (Fig. 3), large areas of elliptical motion (low order resonances) appear at first far from the diagonal, within the regular hyperbolic motion zone. Then, those disappear being converted in higher order resonances. As the latter overlap, that zone becomes filled with chaotic sea too.

The above phase transformations were related to the ray class with the route 100100... . It was believed that any ray exiting within the inner circle can be chosen to be the initial one. Then, arbitrary values of the phase coordinates in the $\mathbb{T}^2_{|11|}$ sheet were assumed by default for trajectories of the type

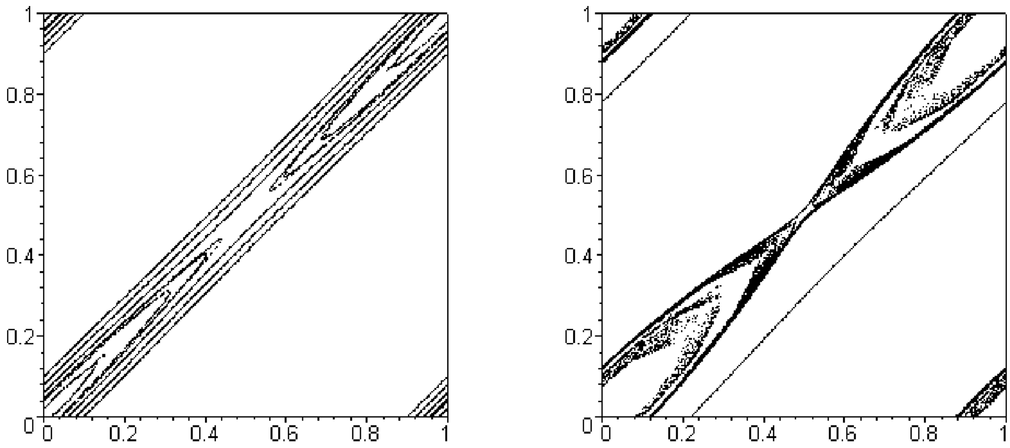

Fig. 2. Phase portrait in the $|11|$ sheet at $\Delta \sim 0.001r$ (left). Typical phase trajectories near the phase space diagonal are presented. As the tore is joined, the areas in the left upper corner and the right lower one presented in the rectangular scan are superposed with its zone close to the diagonal. The phase portrait in the $|11|$ sheet at $\Delta \sim 0.01r$ (right). A chaotic web is manifested near the separatrix being closed together on the tore diagonal. At a distance from the diagonal, one of hyperbolic trajectories is shown the differs insignificantly from the invariant curve of the undisturbed billiard.

mentioned when considering the phase dynamics. At the same time, the $100100\ldots$ route is not admissible for any initial ray when the composite billiard form is distorted. In the symmetric billiard, in contrast, a trajectory with the mentioned route exists always for any initial ray. The prohibition for that simplest evolution type with the ray refraction is connected with the billiard asymmetry and consists in what follows. As a ray is propagated, the sequence of the required route with periodic repetition of 100 cycle is maintained within a certain initial stage. However, at a certain moment, the refracted ray entering the medium 0 may not hit the inner boundary after reflection form the outer one because the billiard is asymmetric (see Fig. 4). Instead, it passes by it and reflects one or more times again from the outer boundary until it hits the inner one and, after the refraction, returns into the medium 1. This delay can be proved to be always finite. For some trajectories, it may be arbitrarily long if the outer and inner billiard boundaries approach arbitrarily to one another, i.e., at $\Delta \to R - r$. After a single reflection in the inner circle, the ray exits again into the outer medium and may undergo again more than one reflection from the outer boundary, and so on. The route of such a trajectory is in general non-periodical and has the form $\left[1001\overbrace{0\cdot\cdot 0}^{n_1}1\overbrace{0\cdot\cdot 0}^{n_2}1\ldots\right]$, where $n_1, n_2, \ldots$ can take any integer values not less than 2. The initial ray can be chosen to be perpendicular to the geometric symmetry axis of the billiard that connects the outer and inner circle centers. Then any trajectory obtained therefrom remains its symmetry with respect to the mentioned axis. In this particular case, it is easy to prove that $n_1, n_2, \ldots$ are always even. The route will take the form $\left[1001\overbrace{(00)\cdot\cdot(00)}^{m_1}1\overbrace{(00)\cdot\cdot(00)}^{m_2}1\ldots\right]$ where $m_1, m_2, \ldots$ are any integers not less than 1. This symmetry is conditioned by the fact that the angular coordinates of the initial ray are interrelated by the equation $S_1 + S_2 = 2\pi$. For example, the fixed point (π, π) on the $\mathbb{T}^2_{|11|}$ diagonal meets that condition. Taking into account the continuity considerations (indices $n_1, n_2, \ldots$ can take discrete values only), the rays close enough to those perpendicular to the billiard axis will have the route identical to the above-mentioned one.

A prolonged residence of a ray in the outer medium prior to its return into the 1 one means a slow filling of the $\mathbb{T}^2_{|11|}$ areas corresponding to the initial rays (in the medium 1) with such routes. Therefore, even in a developed chaos where the most part of the phase space is accessible for any phase trajectory (with any route), the hitting probability into those zones is low enough. In contrast to the rest of the

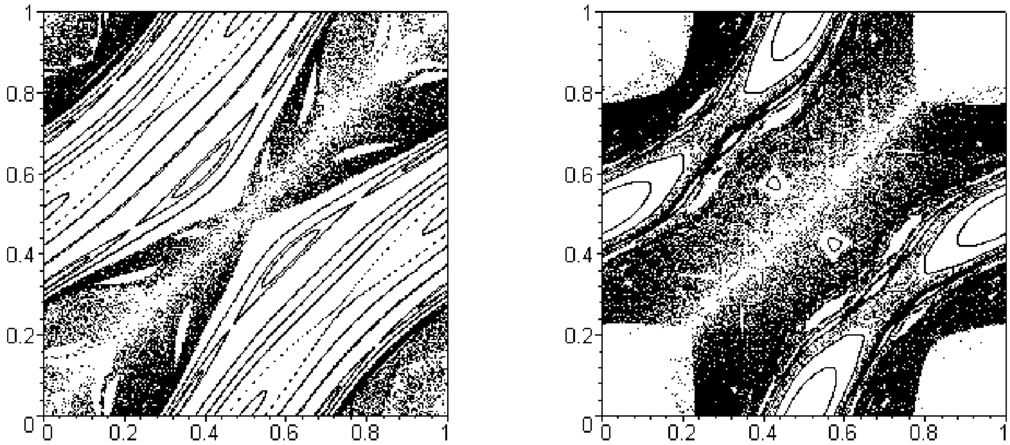

Fig. 3. Phase portrait in the $|11|$ sheet at $\Delta \sim 0.1r$ (left). The phase space becomes mixed. Far from the diagonal, resonances (the elliptic motion zones) arise. Phase portrait in the $|11|$ sheet at $\Delta \sim 0.2r$ (right). Far from the diagonal, high order resonances are formed and overlapped. Near the diagonal, there is a developed chaos. The empty phase areas (quasi lacunas) are also seen near the diagonal, wherefrom the particle is pushed out very fast into the rest of the phase space.

zones, those appear empty in the phase portrait, thus, they can be considered to be quasi lacunas being essentially not visited by a typical phase trajectory. The quasi lacuna volume increases with the billiard asymmetry (Fig. 3). The area of quasi lacunas is seen to correspond to the rays localized more close to the smaller neck between the inner and outer billiard boundaries. Moreover, the diagonal of the $\mathbb{T}^2_{|11|}$ tore is specific in that it reveals the property to form the quasi lacunas to push out the phase trajectories.

Thus, the properties of a system with sufficiently large asymmetry differ considerably from those of undisturbed one. For a symmetric system, the $\mathbb{T}^2_{|00|}$ sheet was not involved at all in the cascade with the ray refraction. It was filled in part due to the reflection cascade only from the outer billiard boundary (as in the usual circular billiard but without reflections from the inner boundary), the other phase space sheets being omitted. None of phase trajectories of that sheet left it at any time. This property is preserved in part at an asymmetry, too. However, new phase trajectories appear in $\mathbb{T}^2_{|00|}$ that come to it or leave it repeatedly. Any ray within the medium 1 can be chosen as the initial one for a subsequent cascade with ray splitting (refraction and reflection). We have proved (the cumbersome calculations are omitted) that there is a critical asymmetry parameter for any initial ray at which the topology of permissible routes permits the repeated visiting of the $|00|$ sheet by the phase trajectories. In this case, the fraction (probability) of trajectories being within the medium 0 is on average larger than that within the medium 1. The numerous calculation show that the trend is intensified as Δ increases and/or $\frac{r}{R}$ or $\frac{n_1}{n_0}$ decrease.

4. Dynamics features under account for total internal reflection

There is another reason for the ray structure complication and the real lacuna appearance within the phase space of a composite billiard. It is connected physically with the total internal reflection phenomenon. For definiteness sake, the refractive index of the inner medium in our asymmetrical billiard is less that that of the outer one, that is, $n_1 < n_0$. Then in the course of ray evolution in the medium 0 it can hit the inner boundary at an angle exceeding the critical one, $\theta_{\mathrm{cr}} = \arcsin\left(\frac{n_1}{n_0}\right)$. In this case, the ray does not pass into a medium of lower density but is reflected back and continues its evolution within the outer medium. Using the angular coordinates chosen by us, the condition for that event can be written as

$$\frac{\rho\left|\sin\left(\tilde{S}_2 - \tilde{S}_1\right)\right|}{\sqrt{\rho^2 + r^2 - 2r\rho\cos\left(\tilde{S}_2 - \tilde{S}_1\right)}} > \frac{n_1}{n_0}\,, \tag{7}$$

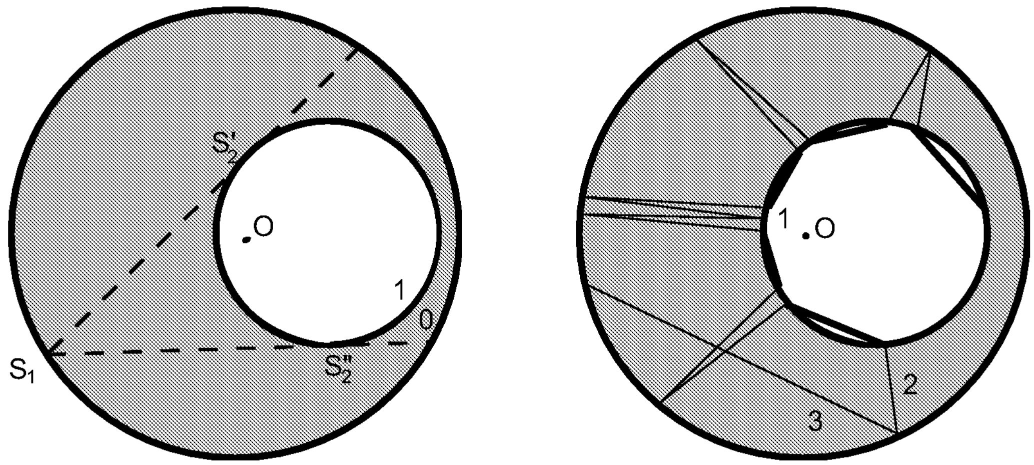

Fig. 4. An illustration of the physical reason for appearance of geometric lacunas (left) and quasi lacunas (right). Right, a trajectory from a quasi lacuna is shown where the route periodical sequence 1001001... is distorted due to multiple reflections from the outer boundary. At first, the ray 1 in the phase space is near the (π,π) point. After the refraction, the ray 2 is reflected from the outer boundary and then the ray 3 does not attain the inner billiard.

where $\tilde{S}_1 = S_1 + \arcsin\left(\frac{\Delta}{\rho}\sin S_1\right)$, $\tilde{S}_2 = S_2 + \arcsin\left(\frac{\Delta}{r}\sin S_2\right)$, $\rho = \sqrt{R^2+\Delta^2-2R\Delta\cos S_1}$. When this condition is met, the phase point enters (in the next iteration) the forbidden area in the corresponding phase sheet $\mathbb{T}^2_{|11|}$, because there is no refracted ray. A plurality of such points forms the lacuna. Similarly, in the case $n_1 > n_0$, a plurality of rays in the inner area (in the previous step of a cascade) with initial coordinates providing the incidence angle thereof to the boundary 1 exceeding the critical value corresponds to a lacuna in the $|01|$ sheet. Note that in a general case, the total internal reflection may result in thinning of the permissible ray routes.

The $|01|$ and $|10|$ sheets are also involved in the complete phase cascade. Since there is no difference in principle between those sheets, let one of them be considered, e.g., $\mathbb{T}^2_{|01|}$. The appearance of the above-considered total internal reflection induced lacunas as well as the geometric lacunas due to the billiard asymmetry is of substantial importance here. The latter lacunas correspond to rays passing the outer medium without reflection from the inner boundary. It is shown, for example, in Fig. 4 that at a fixed position of the ray beginning S_1, only the rays having the second coordinate within $[S_2', S_2]$ interval hit the $|10|$ sheet, all other ones remaining in the $|00|$ sheet. It is noteworthy that their complements in the $\mathbb{T}^2_{|00|}$ sheet forms in turn a lacuna, i.e., the geometric shadow area due to the presence of the inner boundary.

For a symmetric billiard, all the phase sheets are stratified into one-dimensional invariant manifolds [5]. For an asymmetric billiard, all the sheets may include chaotic areas. In $|01|$ and $|10|$ sheets, the chaotic zones are formed most intensely in the areas close to the phase space diagonal, especially near the $(0,0)$ point as we can see on Fig. 4.In the geometric sense, those correspond to the rays incident onto the inner boundary at minimum angles (almost normal), especially in the narrowest site between the inner and outer boundaries. It is just such rays that undergo the maximum angular bundle broadening or scattering of the phase trajectories. In the rest of those sheets, the dynamics is slowed considerably (the phase areas remain empty for a long time), because the visiting thereof is hindered due to total internal reflection effects and/or the presence of geometrical lacunas.

Thus, the work illustrates the characteristic features of the ray picture in billiards with ray splitting, taking a circular asymmetric billiard as an example. In contrast to usual billiards, the fundamental presupposition for the chaotization consists not only in the exponential scattering of the typical phase trajectories (or the bundle angular broadening within almost whole geometric space) but also in the potential unpredictability of the permissible motion routes of rays being subjected to successive refractions at the inner phase interface of the billiard. The billiard geometry effects not only the standard chaotic

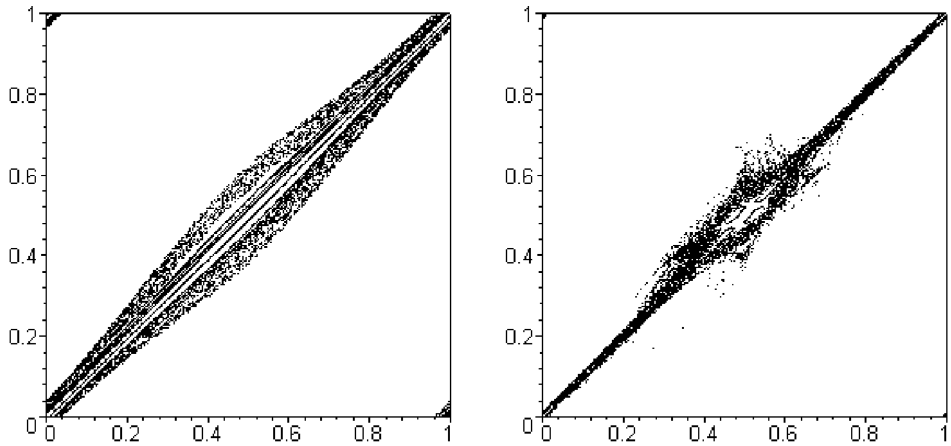

Fig. 5. Phase portrait in the |01| sheet at $\Delta \sim 0.1r$ (left) and $\Delta \sim 0.3r$ (right).

characteristics including the chaotic and regular zone structure within the mixed phase space and reconstructions thereof, the Lyapunov index, etc., but the topology of permissible ray routes. The route thinning occurs due to special geometry as well as to the total internal reflection effect. In the latter case, The trajectory thinning results in the appearance of singular areas of quasi lacunas in the phase space wherefrom the rays are pushed out very fast. The revealed typical features of ray evolution in composite billiards are of importance in studies of light collection effects in "phosphich"detectors consisting of several scintillators with different refraction indices, which are use widely in modern scintillation devices, for example, for positron emission tomography with enhanced space-time resolution.

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Хаотична динаміка асиметричного більярда з заломленням променів

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Досліджено еволюцію променів у композитному асиметричному кільцевому більярді із заломненням променів на внутрішній мезі розділу двлх середовищ. Отриманий кількісний критерій експоненціального кутового розширення довільного пучка променів як необхідна умова хаотизації променевої картини. У фазовому просторі проведено якісний аналіз можливих перебудов фазового портрету вказаного сімейства більярдів при зміні керуючого параметра. Виявлено і пояснено лакунарні особливості фазової динаміки, пов'язані з геометричним проріджуванням допустимих маршрутів променів і з ефектом повного внутрішнього відзеркалення променів від внутрішньої межі більярда. Отримані результати важливі для опису явищ світлозбору у комбінованих детекторах, утворених з декількох різних сцинтиляційних матеріалів.