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Study of the constitutive model of columnar-grained $Cu_{71}Al_{18}Mn_{11}$ shape memory alloy

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The shape memory alloy (SMA) are widely applied in civil construction. among the various SMAs, except Ni-Ti SMAs, Cu-based SMAs(e.g. Cu-Al-Ni, Cu-Zn-Al and Cu-Al-Mn alloys) are the most potential ones for large-scale industrial applications because of their low cost. SMA have good super-elastic and shape memory effect, and be applied in many engineering field very widely. Many scholars study the constitutive model of the SMA material, and build many models to describe relationship between strain and stress. Four constitutive models are introduced detail and some parameters of the equations is determined by formula, those constitutive model include Tanaka constitutive model, Liang-Rogers constitutive model, Brinson constitutive model and Boyd-Lagoudas constitu-tive model. Each constitutive model is induced detail base on thermos-mechanics, thermodynamics and phase transformation dy-namics and some mechanical assumptions. Other constitutive models are introduced also base on the four constitutive models. Those constitutive model of The SMA material is studied to established foundation for further research.

Keywords: constitutive model; shape memory alloy; thermo-mechanics and thermodynamics.

Представлена некоторая конститутивная модель материала столбчато-зернистых сплавов $Cu_{71}Al_{18}Mn_{11}$ с памятью формы (SMA), в том числе конститутивная модель Танаки, конститутивная модель Лян-Роджерса, конститутивная модель Бринсона и конститутивная модель Бойда-Лагудаса. Диаграмма отношений между деформацией и напряжением материала SMA описывается уравнением деформации и напряжения, некоторые параметры этих уравнений вводятся более подробно. Предложенная конститутивная модель материала SMA изучается в качестве основы для дальнейших исследований.

Вивчення конститутивної моделі стовпчато-зернистих сплавів $Cu_{71}Al_{18}Mn_{11}$ з пам'яттю форми. *Liu Wei, Chen Changbing*

Представлено деяку конститутивну модель матеріалу стовпчато-зернистих сплавів $Cu_{71}Al_{18}Mn_{11}$ з пам'яттю форми (SMA), у тому числі конститутивна модель Танаки, конститутивна модель Лян-Роджерса, конститутивна модель Брінсон і конститутивна модель Бойда-Лагудаса. Діаграма відносин між деформацією і напругою матеріалу SMA описується рівнянням деформації і напруги, деякі параметри цих рівнянь вводяться більш докладно. Запропонована конститутивна модель матеріалу SMA вивчається в якості основи для подальших досліджень.

1. Introduction

Because that the shape memory alloy (SMA) material have shape memory effect and hyper-elastic effect, the re-search development and application development of the SMA material are very rapidly in recent years. The device with the SMA material are applied in aeronautical and space field, medical field and civil engineering field. In the field of structural vibration control, the energy dissipation damper with the SMA material are applied in the building structure widely, and have good affect to reduce vibration and isolate vibration.

2. Tanaka Constitutive Model

Base on energy conservation principle and Clausius-Duhem inequalities, Tanaka build constitutive model of the SMA material by basic principle of continuum mechanics [1-3]. The Eq.(1) and Eq.(2) are deduced by the law of thermodynamics below.

$$\rho\dot{U} - \sigma_1 L + \frac{\partial q_{sur}}{\partial \chi} - \rho q = 0, \tag{1}$$

$$\rho\dot{S} - \rho\frac{q}{T} + \frac{\partial}{\partial \chi}\left(\frac{q_{sur}}{T}\right) \ge 0, \tag{2}$$

where ρ is the material density, U is internal energy density, σ_1 is the first kind of Kirchhoff stress, L is velocity gradient, q_{sur} is warm current, χ is material coordinates, q, S are heat source density and entropy density, respectively, T is temperature.

The Eq.(2) is described as Eq.(3) by Helmholtz free energy below.

$$\sigma_2 - \rho_0\frac{\partial\Phi}{\partial\varepsilon}\,\dot{\varepsilon} - \left(S - \frac{\partial\Phi}{\partial T}\right)\dot{T} - \frac{\partial\Phi}{\partial\zeta}\zeta - \frac{q_{sur}}{T\rho}f^{-1}\frac{\partial T}{\partial X} \ge 0, \tag{3}$$

Where σ_2 is the second kind of Kirchhoff stress, Φ is Helmholtz free energy, ρ_0 is relative reference material density, ε is the Green strain, ζ is martensite volume percentage, f is deformation gradient, X is relative reference material coordinates.

Base on thermodynamic theory of continuum mechanics, the coefficient of $\dot{\varepsilon}$ and $\dot{T}$ in Eq.(3) should be zero for the Eq.(2) can be established with any $\dot{\varepsilon}$ and, T Eq.(4) is given below

$$\sigma_2 = \rho_0\frac{\partial\Phi(\varepsilon,\zeta,T)}{\partial\varepsilon} = \sigma(\varepsilon,\zeta,T), \tag{4}$$

The constitutive model Eq.(5) with incremental rates form is given below by differentiating the Eq.(4).

$$\begin{aligned}\dot{\sigma}_2 &= \frac{\partial\sigma}{\partial\varepsilon}\dot{\varepsilon} + \frac{\partial\sigma}{\partial\zeta}\zeta + \frac{\partial\sigma}{\partial T}\dot{T} = \\ &= D(\varepsilon,\zeta,T)\dot{\varepsilon} + \Omega(\varepsilon,\zeta,T)\zeta + \Theta(\varepsilon,\zeta,T)\dot{T}\end{aligned}, \tag{5}$$

where $D(\varepsilon,\zeta,T)\dot{\varepsilon}$ is elastic modulus, $\Omega(\varepsilon,\zeta,T)\zeta$ is phase transition modulus, $\Theta(\varepsilon,\zeta,T)\dot{T}$ is thermo-elastic modulus.

The one dimensional dynamics equation of the martensitic phase transformation is given below by Eq.(6)

$$\begin{cases}\dfrac{\partial\zeta}{1-\zeta} = \alpha^M dT \\ \alpha^M = -\overline{V}Q\,\dfrac{d\Delta G}{dT}\end{cases}, \tag{6}$$

where $\overline{V}$ is average volume of the new martensitic, ΔG is the free driving force when martensitic phase transformation is appearing. Q and α^M are constant respectively.

The martensitic phase transition is given below by Eq.(7) from austenite phase to martensitic phase

$$\zeta = 1 - e^{[\alpha^M(M_S - T) + b^M\sigma]}\ . \tag{7}$$

The austenite phase transition is given below by Eq.(8) from martensitic phase to austenite phase

$$\zeta = 1 - e^{[\alpha^A(A_S - T) + b^A\sigma]}\ , \tag{8}$$

where a^M, a^A, b^M, b^A are constant respectively, the martensitic phase is completed assuming $\zeta = 0.99$, and the austenite phase is completed assuming $\zeta = 0.01$. The four constant are given below by Eq.(9) and Eq.(10)

$$\begin{cases}a^M = \dfrac{\ln(0.01)}{M_s - M_f} \\ a^A = \dfrac{\ln(0.01)}{A_s - A_f}\end{cases}, \tag{9}$$

$$\begin{cases}b^M = \dfrac{a^M}{C^M} \\ b^A = \dfrac{a^A}{C^A}\end{cases}. \tag{10}$$

3. Liang-Rogers Constitutive Model

Base on Tanaka constitutive model, Liang and Rogers build a new constitutive model assuming that the SMA material stress has no relation to loading speed, phase speed and temperature changing speed to get martensitic volume percentage as constitutive model of internal variable [4, 5]. The Liang-Rogers model has clear physical meaning, model parameters are got easily and describe the shape memory effect and super-elastic effect of the SMA material. The Liang-Rogers constitutive model is consistent with the test data goodly, and is classical constitutive model.

Base on thermos-mechanics law, the Liang-Rogers constitutive model of the SMA material is described as Eq.(11) below.

$$\dot{\sigma} = E\dot{\varepsilon} + \theta\dot{T} + \Omega\zeta \ , \quad (11)$$

where $\dot{\sigma}$ and $\dot{\varepsilon}$ are stress speed and strain speed of the SMA material, E is the Young modulus, Ω is phase tensor, θ is thermo-elastic tensor. E and θ are function related to temperature, but the SMA material property is assumed as constant for calculating simply.

The simplification constitutive model of SMA material is describe as Eq.(12) below by differential Eq.(11)

$$\sigma - \sigma_0 = E(\varepsilon - \varepsilon_0) + \Omega(\zeta - \zeta_0) + \theta(T - T_0), \quad (12)$$

where σ is stress, ε is strain, T is temperature, ζ is percentage of martensitic volume that is degree of martensitic phase transformation or reverse phase transformation.

The martensitic phase transformation and martensitic reverse phase transformation are describe by Eq.(13) and Eq.(14) respectively

$$\zeta_{A\to M} = \frac{1-\zeta_A}{2}\cos[a_M(T - M_f) + b_M\sigma] + \frac{1+\zeta_A}{2}, \quad (13)$$

$$\zeta_{M\to A} = \frac{\zeta_M}{2}\cos[a_A(T - A_s) + b_A\sigma] + \frac{\zeta_M}{2}, \quad (14)$$

where a_M, a_A, b_M, b_A, C_M, C_A are described by Eq.(15), Eq.(16) and Eq.(17), respectively.

$$\begin{cases} a_M = \dfrac{\pi}{M_s - M_f} \\ a_A = \dfrac{\pi}{A_f - A_s} \end{cases} \quad (15)$$

$$\begin{cases} b_M = -\dfrac{a_M}{C_M} \\ b_A = -\dfrac{a_A}{C_A} \end{cases} \quad (16)$$

$$\begin{cases} C_M = \tan\alpha \\ C_A = \tan\beta \end{cases}, \quad (17)$$

where ζ_A, ζ_M are percentage of martensitic at the beginning of phase transformation and reverse phase transformation respectively.

In this paper, the super-elastic mechanics property of the SMA material is studied under the condition of constant temperature, so the Liang-Rogers constitutive model is super-elastic mechanics model of the SMA material under the condition of constant temperature.

Assume that the initial stress and initial strain are zero, $\sigma_0 = 0$, $\varepsilon_0 = 0$, and $T > A_f$, the SMA material is completely austenite and $\zeta_A = 0$. Assume that temperature is constant under tensile condition of the constant temperature, $T = T_0$. The Eq.(18) is given below by rewriting the Eq.(12).

$$\sigma = E\varepsilon + \Omega\zeta \ . \quad (18)$$

When starting loading, the SMA material is condition of austenite completely and the relationship is linear between strain and stress, $\sigma = E\varepsilon$.

The SMA stress is induced martensitic phase transformation, when the stress increase to σ_a. The yield stress σ_a and the yield strain ε_a are given below by bringing into Eq.(4) with $\zeta_A = 0$.

$$\begin{cases} \sigma_a = C_M(T - M_s) \\ \varepsilon_a = \dfrac{\sigma_a}{E_A} \end{cases} . \quad (19)$$

The SMA stress is martensitic phase transformation, when the stress increase to $\zeta = 1$. The yield stress σ_b and the yield strain ε_b are given below by bringing into Eq.(4) with $\zeta = 1$.

$$\begin{cases} \sigma_b = C_M(T - M_f) \\ \varepsilon_b = -\dfrac{C_M(T - M_f)}{E} + \varepsilon_L \end{cases}, \quad (20)$$

where ε_L is maximum recovery strain of the SMA material. when the stress is unloading to σ_c, the martensitic phase transformation is happening, $\zeta = 1$, σ_d and ε_d are given by Eq.(21) below

$$\begin{cases} \sigma_d = C_A(T - A_f) \\ \varepsilon_d = \dfrac{\sigma_d}{E_A} \end{cases} . \quad (21)$$

After martensitic phase transformation, the SMA material is not martensitic phase transformation any more, and the relationship is linear between strain and stress, $\sigma = E_A\varepsilon$.

When the super-elastic constitutive model of the SMA material is phase transformation, the relationship between stress and strain is determined by inverse cosine function, and the calculated process is very complex. Liang and Rogers linear relationship between stress and strain in process of phase transformation by character of phase transformation to get simplified SMA material constitutive model of the super-elastic. The simplified tension constitutive model is given by Eq.(22) and Eq.(23) below.

Loading condition.

$$\sigma = \begin{cases} E_A\varepsilon & (\varepsilon \le \varepsilon_a) \\ E_A\varepsilon_a + E_{ab}(\varepsilon - \varepsilon_a) & (\varepsilon_a < \varepsilon < \varepsilon_b) \end{cases}, \quad (22)$$

Unloading condition.

$$\sigma = \begin{cases} \sigma_b + E_A(\varepsilon - \varepsilon_b) & (\varepsilon_c < \varepsilon < \varepsilon_b) \\ E_A\varepsilon_d + E_{cd}(\varepsilon - \varepsilon_d) & (\varepsilon_a < \varepsilon < \varepsilon_b), \\ E_A\varepsilon & (\varepsilon \le \varepsilon_d) \end{cases} \quad (23)$$

$$\begin{cases} E_{ab} = \dfrac{C_M(M_s - M_f)}{\varepsilon_b - \varepsilon_a} \\ E_{cd} = \dfrac{C_A(A_f - A_s)}{\varepsilon_c - \varepsilon_d} \end{cases}, \quad (24)$$

where σ is stress, ε is strain, E is Young modulus, E_{ab} and E_{cd} are slope of line ab and line cd of the linear hysteretic model in process of super-elastic.

4. Brinson Constitutive Model

Base on Tanaka and Liang-Rogers constitutive model, Brinson build a new SMA material constitutive model and adopt non-constant material parameters to divide martensitic volume percentage into two part which be given by Eq.(25) below[6]

$$\zeta = \zeta_T + \zeta_S , \quad (25)$$

where ζ_T is martensitic volume percentage caused by temperature, ζ_S is martensitic volume percentage caused by stress.

Brinson think that elastic modulus and phase transformation are linear relationship with martensitic volume percentage, the one dimensional constitutive model of Brinson is describe as incremental rates by Eq.(26) and Eq.(27) below

$$D = D_A + \zeta(D_M - D_A) , \quad (26)$$

$$\Omega = -\varepsilon_L \cdot D , \quad (27)$$

where D_A is elastic modulus of austenite condition, D_M is elastic modulus of martensitic condition.

The SMA material constitutive model is given by Eq.(29) with SMA material residual strain formula Eq.(28)[7]

$$\varepsilon_{res} = \varepsilon_L \cdot \zeta_S , \quad (28)$$

$$\sigma - \sigma_0 = D(\zeta)\varepsilon - D(\zeta_0)\varepsilon_0 + \Omega(\zeta)\zeta_S - \Omega(\zeta_0)\zeta_{S0} + \Theta(T - T_0) . \quad (29)$$

Fig.1 show us the relationship diagram between phase transformation stress and phase transformation temperatures of one dimensional SMA material constitutive model from lot of tests and theory analysis.

Brinson amendment furtherly relationship between critical stress and martensitic phase transformation temperature to build motion control equation of the SMA material phase transformation.

The process of martensitic phase transformation are given by Eq.(30) and Eq.(31) below.

$$\zeta_S = \frac{1 - \zeta_{S0}}{2}\cos\left\{\frac{\pi}{\sigma_s^{cr} - \sigma_f^{cr}}[\sigma - \sigma_f^{cr} - C_M(T - M_S)]\right\} + \frac{1 + \zeta_{S0}}{2} \quad (30)$$

$$\zeta_T = \zeta_{T0} - \frac{\zeta_{T0}}{1 - \zeta_{S0}}(\zeta_S - \zeta_{S0}) \quad (31)$$

The process of austenite phase transformation are given by Eq.(32) and Eq.(33) below.

$$\zeta_S = \zeta_{S0} - \frac{\zeta_{S0}}{\zeta_0}\left\{\zeta_{S0} - \frac{\zeta_0}{2}\left[\cos\left(\frac{C_AT - C_AA_S - \sigma}{C_AA_f - C_AA_S}\pi\right) + 1\right]\right\}, \quad (32)$$

$$\zeta_T = \zeta_{T0} - \frac{\zeta_{T0}}{\zeta_0}\left\{\zeta_0 - \frac{\zeta_0}{2}\left[\cos\left(\frac{C_AT - C_AA_S - \sigma}{C_AA_f - C_AA_S}\pi\right) + 1\right]\right\}. \quad (33)$$

The jump function is applied in Eq.(30), that is given by Eq.(34) below.

$$(x - a)_+^n = \begin{cases} (x - a)^n & x > a \\ 0 & x \le a \end{cases} \quad (34)$$

Where σ_s^{cr} is the critical stress at beginning of martensitic phase transformation, σ_f^{cr} is the critical stress at end of martensitic phase transformation.

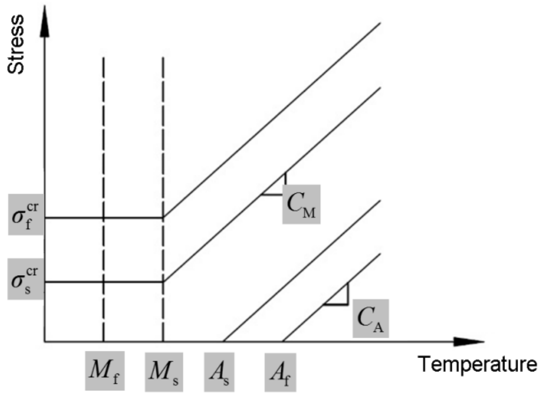

Fig. 1. Relationship diagram of Brinson constitutive model between phase transformation stress and phase transformation temperatures

5. The Other Constitutive Models

Tanaka model, Liang-Rogers model and Brinson model are applied very widely and have strong engineering applicability, the three model is build base on thermodynamics, thermos-mechanics and phase transformation theory of the SMA material. The other scholars also study other constitutive model of the SMA material base on thermodynamics, thermos-mechanics and phase transformation theory of the SMA material.

Base on Tanaka constitutive model and Liang-Rogers constitutive model and assuming that the shape memory effect of the SMA material is similar to yield effect of the isotropic material, the Boyd-Lagoudas constitutive model is given with Eq.(35),Eq.(36) and Eq.(37) by using equivalent stress instead of one dimensional stress of the SMA material. The Boyd-Lagoudas constitutive model is given by Eq.(35),Eq.(36) and Eq.(37) below.

$$\sigma_{ij} = D_{ijkl}[\varepsilon_{kl} - \varepsilon_{kl}^{tr} - \alpha_{kl}(T - T_0)], \quad (35)$$

$$\dot{\varepsilon}_{kl}^{tr} = \lambda_{kl}\zeta , \quad (36)$$

$$\lambda_{kl} = \begin{cases} -\dfrac{3\Omega S_{kl}}{2D\ \overline{\sigma}} & \zeta>0 \\ -\dfrac{\Omega \varepsilon_{kl}^{tr}}{D\overline{\varepsilon}^{tr}} & \zeta<0 \end{cases}, \quad (37)$$

where ε_{kl}^{tr} is phase strain, $\overline{\varepsilon}^{tr}$ is equivalent phase strain, S_{kl} is deviatoric stress, $\overline{\sigma}$ is equivalent stress, α_{kl} is thermal expansion coefficient of the SMA material.

The Boyd-Lagoudas constitutive model describe SMA material phase transformation by martensitic volume percentage as same as Tanaka model and Liang-Rogers model, but the Boyd-Lagoudas constitutive model is three dimensional constitutive model.

Ivshin and Pence build Ivshin-Pence constitutive model base on thermo-mechanics theory and thermodynamics theory. The Ivshin-Pence constitutive model get austenite volume percentage as internal variable, and Ivshin-Pence constitutive model is given by Eq.(38) and Eq.(39) below

$$\varepsilon = (1 - \alpha)\varepsilon_M + \alpha\varepsilon_A \quad (38)$$

$$\begin{cases} \varepsilon_A = \dfrac{\sigma}{E_A} \\ \varepsilon_M = \dfrac{\sigma}{E_M} + \varepsilon_L \end{cases} \quad (39)$$

6. Conclusions

In the paper, some constitutive model of the SMA material are introduced detail, including Tanaka constitutive model, Liang-Rogers constitutive model, Brinson constitutive model and Boyd-Lagoudas constitutive model. The relationship diagram between strain and stress of the SMA material is described by equation of strain and stress and some parameters of those equations are introduced detail. Those constitutive models also describe condition from martensitic phase to austenite phase.

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